

Bounds on Quantum Multiple-Parameter Estimation with Gaussian State

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We investigate the quantum Cramer-Rao bounds on the joint multiple-parameter estimation with the Gaussian state as a probe. We derive the explicit right logarithmic derivative and symmetric logarithmic derivative operators in such a situation. We compute the corresponding quantum Fisher information matrices, and find that they can be fully expressed in terms of the mean displacement and covariance matrix of the Gaussian state. Finally, we give some examples to show the utility of our analytical results.

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I. INTRODUCTION

Employing quantum resources to improve the sensitivity in the estimation of relevant physical parameters is of great importance in metrology and sensing [1]. Much of the work has focused on the estimation of a single parameter, both theoretically and experimentally [1, 2]. However, there are many situations where the joint estimation of multiple parameters becomes necessary, e.g., the vector phase estimation in recent field of phase imaging and microscopy [3]. The developed multi-port devices and multi-qubit manipulation also demand the investigation of multi-parameter sensitivity from theoretical viewpoint.

The joint estimation of multiple parameters is an example of the general problem of quantum estimation theory [4]. A typical parameter estimation consists in sending a probe in a suitable initial state through some parameter-dependent physical process and measuring the final state of the probe, estimating then from this measurement the values of the parameters. Consider a family of quantum state ρ_θ which depend on a set of d different parameters $\theta = (\theta_1, \dots, \theta_d)^T$. The aim of quantum estimation theory is to infer the values of θ from the outcomes of a generalized measurement M (characterized by POVM $\{M_\xi\}$, $\xi = (\xi_1, \xi_2, \dots)^T$, $M_\xi \geq 0$, $\int d\xi M_\xi = \mathbb{1}$). Let $\Theta(\xi)$ be the estimator of θ constructed from the outcome ξ . To quantify the sensitivity of this estimation, a local covariance matrix is defined as $V_\theta(M) = \int d\xi (\Theta(\xi) - \theta)(\Theta(\xi) - \theta)^T p(\xi|\theta)$, where $p(\xi|\theta) = \text{Tr}[\rho_\theta M_\xi]$ is the conditional probability distribution of obtaining a certain outcome ξ given θ . From now on we assume a particular point θ and consistently drop the dependency on θ .

In order to present the lower bounds for $V(M)$, one can define the so-called right logarithmic derivative (RLD) and symmetric logarithmic derivative (SLD) operators

for each of the parameters involved [4], respectively as

$$\partial_k \rho = \rho \mathcal{L}_k \quad (\text{RLD}), \quad (1)$$

$$\partial_k \rho = \frac{1}{2}(\rho \mathcal{L}_k + \mathcal{L}_k \rho) \quad (\text{SLD}), \quad (2)$$

where $\partial_k \equiv \partial/\partial\theta_k$. Then one can define two matrices

$$\mathcal{F}_{ij} = \text{Tr}[\rho \mathcal{L}_i \mathcal{L}_j^\dagger] = \mathcal{F}_{ji}^*, \quad (3)$$

$$\mathcal{F}_{ij} = \frac{1}{2} \text{Tr}[\rho(\mathcal{L}_i \mathcal{L}_j + \mathcal{L}_i \mathcal{L}_j)] = \mathcal{F}_{ji}, \quad (4)$$

which are called the RLD and SLD quantum Fisher information (QFI) matrices, respectively. The SLD QFI matrix can be computed from Uhlmann's quantum fidelity between two outgoing final states corresponding to two different sets of parameters [5]. By defining a positive definite matrix G , two different Cramer-Rao bounds hold [4],

$$\text{Tr}[GV(M)] \geq \frac{1}{\nu} (\text{Tr}[G\mathcal{F}_R^{-1}] + \text{Tr}[G\mathcal{F}_I^{-1}]), \quad (5)$$

$$\text{Tr}[GV(M)] \geq \frac{1}{\nu} \text{Tr}[G\mathcal{F}^{-1}], \quad (6)$$

where $|A| \equiv \sqrt{AA^\dagger}$ and ν is the number of the measurements performed. If we choose $G = \mathbb{1}$, we obtain the two bounds on the sum of the variances of the parameters involved,

$$\nu \sum_{k=1}^d \delta^2 \theta_k \geq B_R \equiv \text{Tr}[\mathcal{F}_R^{-1}] + \text{Tr}[\mathcal{F}_I^{-1}], \quad (7)$$

$$\nu \sum_{k=1}^d \delta^2 \theta_k \geq B_S \equiv \text{Tr}[\mathcal{F}^{-1}]. \quad (8)$$

In general, neither the RLD bound nor the SLD bound is attainable [6]. Here the fundamental non-commutativity of quantum theory forbids simultaneously obtaining the optimal estimations of all parameters, and optimizing the measurement for one parameter will usually disturb the measurement precision on the others. At the same time, the optimal estimator for the RLD bound

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might not correspond to a POVM. On the other hand, even if the optimal measurements for both parameters do not commute, it is still possible to attain both bounds by devising a single simultaneous measurement.

Most of the work on quantum parameter estimation are devoted to the SLD bound [6–9] (notable contributions are Refs. [10, 11]). For single-parameter estimation problems, it is known that the SLD QFI is always smaller than the RLD QFI, and thus gives a tighter bound for parameter sensitivity [4]. Moreover, the single-parametric SLD QFI is asymptotically attainable for large N . For multi-parametric cases, recent progress in the theory of local asymptotic normality for quantum states suggests that Eq. (6) is asymptotically attainable if and only if

$$\text{Tr}[\rho[\mathcal{L}_i, \mathcal{L}_j]] = 0. \quad (9)$$

When this condition can not be fulfilled, the RLD bound could be tighter, and therefore becomes more important.

The analytical expressions for the RLD and SLD operators and the QFI matrices always pose formidable challenges, except for very particular situations when the density state can be simply put in the diagonal form [2, 9]. For the Gaussian state, the author of Ref. [8] has obtained the explicit forms of the SLD operator and the corresponding QFI matrix in terms of the mean displacement and covariance matrix. In this paper, we generalized it to the RLD case and derive the relevant results.

We organize this paper as follows. Section II reviews some basic notations for the Gaussian state. Then in section III we derive the expressions for the RLD and SLD operators and the QFI matrices for multi-parameter estimation. In section IV we give some examples to show the utility of our obtained results. Finally, we end with a short summary.

II. NOTATIONS FOR GAUSSIAN STATE

Consider n bosonic modes with annihilation operators a_i satisfying the commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$, all other commutators being zero. Arrange all operators into a vector $a^\mu = (a_1, a_1^\dagger, a_2, a_2^\dagger, \dots)^T$. From now on Einstein's summation convention will be used throughout this paper. The commutation relations are expressed as $[a^\mu, a^\nu] = \Omega^{\mu\nu}$ and $a^{\mu\dagger} = X_\nu^\mu a^\nu \equiv X^{\mu\nu} a^\nu$, where $\Omega = \bigoplus_{k=1}^n i\sigma_y$ and $X = \bigoplus_{k=1}^n \sigma_x$, in terms of the x, y -component Pauli matrices $\sigma_{x,y}$. Note that $\Omega^T = -\Omega$ and $\Omega^2 = -\mathbb{1}$. Let us introduce the convention $\Omega_{\mu\nu} = \Omega^{\mu\nu}$, so that $\Omega^{\alpha\beta}\Omega_{\beta\gamma} = -\delta_\gamma^\alpha$, to lower (rise) the index of a^μ (a_μ) through $a_\mu = \Omega_{\mu\nu} a^\nu$ and $a^\mu = \Omega^{\nu\mu} a_\nu$. Equivalently, the position and momentum operators are arranged in the vector $x^\mu = (q_1, p_1, q_2, p_2, \dots)^T$, related to a^μ by a unitary transformation $x^\mu = H_\nu^\mu a^\nu \equiv H^{\mu\nu} a_\nu$, where

$$H = \bigoplus_{k=1}^n \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad H^\dagger H = 1. \quad (10)$$

The Gaussian state is defined as a state with Gaussian characteristic function

$$\chi(z) = \text{Tr}[\rho e^{-z_\mu a^\mu}] = \exp\left(\frac{1}{2}\Sigma^{\mu\nu} z_\mu z_\nu - \lambda^\mu z_\mu\right), \quad (11)$$

which is fully described by the mean displacement and covariance matrix

$$\lambda^\mu = \text{Tr}[\rho a^\mu] \quad (12)$$

$$\Sigma^{\mu\nu} = \frac{1}{2}\text{Tr}[\rho(\tilde{a}^\mu \tilde{a}^\nu + \tilde{a}^\nu \tilde{a}^\mu)] \quad (13)$$

in terms of the centered operator $\tilde{a}^\mu = a^\mu - \lambda^\mu$. Here $z^\mu = (z_1, z_1^*, z_2, z_2^*, \dots)^T$, $z^{\mu*} = X_\nu^\mu z^\nu$, and $z_\mu = \Omega_{\mu\nu} z^\nu$.

With the above notations, the following identities will be frequently used,

$$X_\nu^\mu \tilde{a}^\nu = \tilde{a}^{\mu\dagger} \quad (14)$$

$$X_\nu^\mu \lambda^\nu = \lambda^{\mu*} \quad (15)$$

$$\text{Tr}[\rho \tilde{a}^\mu \tilde{a}^\nu] = \Sigma_+^{\mu\nu} \quad (16)$$

$$X_\gamma^\mu X_\delta^\nu \Sigma^{\gamma\delta} = \Sigma^{\mu\nu*} \quad (17)$$

$$X_\gamma^\mu X_\delta^\nu \Omega^{\gamma\delta} = -\Omega^{\mu\nu} \quad (18)$$

$$\Sigma^{\mu\nu} = \Sigma^{\nu\mu} \quad (19)$$

$$\Omega^{\mu\nu} = -\Omega^{\nu\mu} \quad (20)$$

$$\Sigma_+^{\mu\nu} = \Sigma_-^{\nu\mu}, \quad (21)$$

where $\Sigma_\pm \equiv \Sigma \pm \Omega/2$.

III. DERIVATION OF THE RLD AND SLD OPERATORS

In this section, we derive the RLD and SLD operators for the Gaussian state. For simplicity, we first consider the RLD case. It is known [12] that a displaced gaussian state is related to a centered gaussian state by $\rho_\lambda = D(\lambda)\rho_0 D^\dagger(\lambda)$, where the unitary matrix $D(\lambda) \equiv e^{\lambda^\mu a_\mu}$ and $D a^\mu D^\dagger = \tilde{a}^\mu$. The right hand side of Eq. (1) can be written as

$$\partial_k \rho_\lambda = \partial'_k \rho_\lambda + D(\partial_k \rho_0) D^\dagger, \quad (22)$$

where the symbol ∂'_k in the first term means only taking the derivative of λ^μ with respect to θ_k , and the second term corresponds to the RLD for the centered state.

For the first term, since the RLD of a Gaussian state with respect to θ_k of λ^μ involves the data linearly, we try the linear form $\mathcal{L}'_k = \mathcal{B}_\mu^{(k)} a^\mu + \mathcal{C}'_k$. Then we convert Eq. (1) into c-number equation through the characteristic function and the operator rule [12], $\rho a^\mu \rightarrow (-\partial^\mu + z^\mu/2)\chi$, where $\partial^\mu \equiv \partial/\partial z_\mu$. This leads to

$$\begin{aligned} \rho a^\mu &\rightarrow \left(-\Sigma^{\mu\nu} z_\nu + \lambda^\mu + \frac{1}{2}\Omega^{\nu\mu} z_\nu\right)\chi \\ &\equiv (-\Sigma_-^{\mu\nu} z_\nu + \lambda^\mu)\chi \end{aligned} \quad (23)$$

and Eq. (1) implies

$$\Sigma_{-}^{\nu\mu} \mathcal{B}_{\mu}^{(k)} = \partial_k \lambda^{\nu}, \quad (24)$$

$$\mathcal{B}_{\mu}^{(k)} \lambda^{\mu} + \mathcal{C}'_k = 0. \quad (25)$$

Next we consider $\partial_k \rho_0$ in the second term of Eq. (22). Since the RLD of a centered Gaussian state with respect to θ_k of $\Sigma^{\mu\nu}$ involves the data quadratically, we try the quadratic form $\mathcal{L}_k'' = \mathcal{A}_{\mu\nu}^{(k)} a^{\mu} a^{\nu} + \mathcal{C}_k''$ with $\mathcal{A}_{\mu\nu}^{(k)} = \mathcal{A}_{\nu\mu}^{(k)}$. We also convert Eq. (1) into c-number equation via the operator rule,

$$\begin{aligned} \rho a^{\mu} a^{\nu} &\rightarrow \left[\partial^{\mu} \partial^{\nu} + \frac{1}{2} \Omega^{\mu\nu} + \frac{1}{4} \Omega^{\alpha\mu} \Omega^{\beta\nu} z_{\alpha} z_{\beta} \right. \\ &\quad \left. - \frac{1}{2} (\Omega^{\alpha\nu} \partial^{\mu} + \Omega^{\alpha\mu} \partial^{\nu}) z_{\alpha} \right] \chi \\ &= \left(\Sigma^{\mu\nu} + \frac{1}{2} \Omega^{\mu\nu} \right) \chi + \left[\Sigma^{\alpha\mu} \Sigma^{\beta\nu} + \frac{1}{4} \Omega^{\alpha\mu} \Omega^{\beta\nu} \right. \\ &\quad \left. - \frac{1}{2} (\Sigma^{\alpha\mu} \Omega^{\beta\nu} + \Omega^{\alpha\mu} \Sigma^{\beta\nu}) \right] z_{\alpha} z_{\beta} \chi \\ &= \Sigma_{+}^{\mu\nu} \chi + \mathbf{M}^{\alpha\beta, \mu\nu} z_{\alpha} z_{\beta} \chi, \end{aligned} \quad (26)$$

where $\mathbf{M} \equiv \Sigma_{-} \otimes \Sigma_{-}$. So Eq. (1) implies

$$\mathbf{M}^{\alpha\beta, \mu\nu} \mathcal{A}_{\mu\nu}^{(k)} = \frac{1}{2} \partial_k \Sigma^{\alpha\beta}, \quad (27)$$

$$\mathcal{A}_{\mu\nu}^{(k)} \Sigma_{+}^{\mu\nu} + \mathcal{C}_k'' = 0. \quad (28)$$

Combining the above results, the RLD takes

$$\begin{aligned} \mathcal{L}_k &= \tilde{\mathcal{L}}_k'' + \mathcal{L}_k' = D \mathcal{L}_k'' D^{\dagger} + \mathcal{L}_k' \\ &= \mathcal{A}_{\mu\nu}^{(k)} (\tilde{a}^{\mu} \tilde{a}^{\nu} - \Sigma^{\mu\nu}) + \mathcal{B}_{\mu}^{(k)} \tilde{a}^{\mu}, \end{aligned} \quad (29)$$

where

$$\mathcal{A}_{\mu\nu}^{(k)} = \frac{1}{2} \mathbf{M}_{\mu\nu, \alpha\beta}^{-1} \partial_k \Sigma^{\alpha\beta}, \quad (30)$$

$$\mathcal{B}_{\mu}^{(k)} = (\Sigma_{-})_{\mu\nu}^{-1} \partial_k \lambda^{\nu}. \quad (31)$$

Then, using the Wick theorem for the Gaussian state [15], we compute the RLD QFI matrix. Since the odd terms of the cross products of $\tilde{\mathcal{L}}_k''$ and \mathcal{L}_k' vanish identically, we only consider the terms involving the double contributions of $\tilde{\mathcal{L}}_k''$ and \mathcal{L}_k' , respectively. Noticing

$$\text{Tr}[\rho \tilde{a}^{\alpha} \tilde{a}^{\beta} \tilde{a}^{\mu} \tilde{a}^{\nu}] = \Sigma_{+}^{\alpha\beta} \Sigma_{+}^{\mu\nu} + \Sigma_{+}^{\alpha\mu} \Sigma_{+}^{\beta\nu} + \Sigma_{+}^{\alpha\nu} \Sigma_{+}^{\beta\mu}, \quad (32)$$

we have

$$\begin{aligned} &\text{Tr}[\rho \tilde{\mathcal{L}}_i'' \tilde{\mathcal{L}}_j''^{\dagger}] \\ &\stackrel{(16,14)}{=} \mathcal{A}_{\alpha\beta}^{(i)} \mathcal{A}_{\mu\nu}^{(j)*} (X_{\gamma}^{\mu} X_{\delta}^{\nu} \text{Tr}[\rho \tilde{a}^{\alpha} \tilde{a}^{\beta} \tilde{a}^{\gamma} \tilde{a}^{\delta}] - \Sigma^{\alpha\beta} \Sigma^{\mu\nu*}) \\ &\stackrel{(17,32)}{=} \mathcal{A}_{\alpha\beta}^{(i)} \mathcal{A}_{\mu\nu}^{(j)*} X_{\gamma}^{\mu} X_{\delta}^{\nu} \left(\Sigma_{+}^{\alpha\gamma} \Sigma_{+}^{\beta\delta} + \Sigma_{+}^{\alpha\delta} \Sigma_{+}^{\beta\gamma} \right) \\ &\stackrel{(21)}{=} X_{\gamma}^{\mu} X_{\delta}^{\nu} \left(\Sigma_{-}^{\gamma\alpha} \Sigma_{-}^{\delta\beta} + \Sigma_{-}^{\delta\alpha} \Sigma_{-}^{\gamma\beta} \right) \mathcal{A}_{\alpha\beta}^{(i)} \mathcal{A}_{\mu\nu}^{(j)*} \\ &\stackrel{(27)}{=} \frac{1}{2} X_{\gamma}^{\mu} X_{\delta}^{\nu} (\partial_i \Sigma^{\gamma\delta} + \partial_i \Sigma^{\delta\gamma}) \mathcal{A}_{\mu\nu}^{(j)*} \\ &\stackrel{(17,19)}{=} (\partial_i \Sigma^{\mu\nu*}) \mathcal{A}_{\mu\nu}^{(j)*} \stackrel{(3)}{=} (\partial_j \Sigma^{\mu\nu}) \mathcal{A}_{\mu\nu}^{(i)} \\ &\stackrel{(30)}{=} \frac{1}{2} (\partial_j \Sigma^{\mu\nu}) \mathbf{M}_{\mu\nu, \alpha\beta}^{-1} \partial_i \Sigma^{\alpha\beta}, \end{aligned} \quad (33)$$

where the number over the equal sign means the corresponding equation has been used. On the other hand,

$$\begin{aligned} &\text{Tr}[\rho \mathcal{L}_i' \mathcal{L}_j'^{\dagger}] \stackrel{(14)}{=} \mathcal{B}_{\alpha}^{(i)} \mathcal{B}_{\mu}^{(j)*} X_{\gamma}^{\mu} \text{Tr}[\rho \tilde{a}^{\alpha} \tilde{a}^{\gamma}] \\ &\stackrel{(16)}{=} X_{\gamma}^{\mu} \Sigma_{+}^{\alpha\gamma} \mathcal{B}_{\alpha}^{(i)} \mathcal{B}_{\mu}^{(j)*} \stackrel{(21)}{=} X_{\gamma}^{\mu} \Sigma_{-}^{\gamma\alpha} \mathcal{B}_{\alpha}^{(i)} \mathcal{B}_{\mu}^{(j)*} \\ &\stackrel{(24)}{=} X_{\gamma}^{\mu} (\partial_i \lambda^{\gamma}) \mathcal{B}_{\mu}^{(j)*} \stackrel{(15)}{=} (\partial_i \lambda^{\mu*}) \mathcal{B}_{\mu}^{(j)*} \\ &\stackrel{(3)}{=} (\partial_j \lambda^{\mu}) \mathcal{B}_{\mu}^{(i)} \stackrel{(31)}{=} (\partial_j \lambda^{\mu}) (\Sigma_{-}^{-1})_{\mu\nu} (\partial_i \lambda^{\nu}). \end{aligned} \quad (34)$$

The final expression for the RLD QFI matrix is thus obtained

$$\mathcal{F}_{ij} = \frac{1}{2} \mathbf{M}_{\alpha\beta, \mu\nu}^{-1} \partial_j \Sigma^{\alpha\beta} \partial_i \Sigma^{\mu\nu} + (\Sigma_{-})_{\mu\nu}^{-1} \partial_j \lambda^{\mu} \partial_i \lambda^{\nu}. \quad (35)$$

According to Ref. [4], the RLD bound is attainable if there is a POVM $\{M_{\hat{\theta}}\}$, so that for a particular vector \mathbf{Y} , the following equation

$$\sum_k (\mathbf{Y}^{\dagger} \mathcal{F}^{-1})_k \mathcal{L}_k M_{\hat{\theta}} = c \sum_k (\hat{\theta}_k - \theta_k) M_{\hat{\theta}} \quad (36)$$

holds for some complex number c , which may be a function of θ . The specific case for the RLD bound of estimating the parameters of a coherent signal in thermal background has been obtain in Ref. [4].

Similarly using the rules $(\rho a^{\mu} + a^{\mu} \rho)/2 \rightarrow \partial^{\mu} \chi$ and

$$\frac{1}{2} (\rho a^{\mu} a^{\nu} + a^{\mu} a^{\nu} \rho) \rightarrow \left[\partial^{\mu} \partial^{\nu} + \frac{1}{2} \Omega^{\mu\nu} + \frac{1}{4} \Omega^{\alpha\mu} \Omega^{\beta\nu} z_{\alpha} z_{\beta} \right] \chi,$$

the SLD takes

$$\mathcal{L}_k = \mathcal{A}_{\mu\nu}^{(k)} (\tilde{a}^{\mu} \tilde{a}^{\nu} - \Sigma^{\mu\nu}) + \mathcal{B}_{\mu}^{(k)} \tilde{a}^{\mu}, \quad (37)$$

where in terms of $\mathfrak{M} \equiv \Sigma \otimes \Sigma + \Omega \otimes \Omega/4$,

$$\mathcal{A}_{\mu\nu}^{(k)} = \frac{1}{2} \mathfrak{M}_{\mu\nu, \alpha\beta}^{-1} \partial_k \Sigma^{\alpha\beta}, \quad (38)$$

$$\mathcal{B}_{\mu}^{(k)} = \Sigma_{\mu\nu}^{-1} \partial_k \lambda^{\nu}, \quad (39)$$

and

$$\mathcal{F}_{ij} = \frac{1}{2} \mathfrak{M}_{\alpha\beta, \mu\nu}^{-1} \partial_j \Sigma^{\alpha\beta} \partial_i \Sigma^{\mu\nu} + \Sigma_{\mu\nu}^{-1} \partial_j \lambda^{\mu} \partial_i \lambda^{\nu}. \quad (40)$$

Eqs. (29, 35, 37, 40) are the main results of our paper. For the SLD case, they are identical with the results in Ref. [8] obtained by a different method.

Finally, we consider the asymptotic attainability of the SLD bound. Using the relations $\text{Tr}[\rho[\tilde{a}^{\alpha}, \tilde{a}^{\beta}]] = \Omega^{\alpha\beta}$ and

$$\begin{aligned} \text{Tr}[\rho[\tilde{a}^{\alpha} \tilde{a}^{\beta}, \tilde{a}^{\mu} \tilde{a}^{\nu}]] &= \Sigma_{+}^{\alpha\mu} \Omega^{\beta\nu} + \Sigma_{+}^{\alpha\nu} \Omega^{\beta\mu} \\ &\quad + \Sigma_{+}^{\mu\beta} \Omega^{\alpha\nu} + \Sigma_{+}^{\nu\beta} \Omega^{\alpha\mu}, \end{aligned} \quad (41)$$

the condition (9) can be simply put in the form

$$4 \mathcal{A}_{\alpha\beta}^{(i)} \mathcal{A}_{\mu\nu}^{(j)} \Sigma^{\alpha\mu} \Omega^{\beta\nu} + \mathcal{B}_{\alpha}^{(i)} \mathcal{B}_{\beta}^{(j)} \Omega^{\alpha\beta} = 0. \quad (42)$$

IV. APPLICATIONS

In this section, we take the Gaussian state as a probe to consider the estimation of the single phase, the two conjugate parameters in the displacement operator, the damping and temperature of a bosonic channel, and the squeezing and phase in the squeezing operator.

A. Phase estimation with two-mode squeezed vacuum

First, we consider the lossy quantum optical metrology with a two-mode (a_1, a_2) squeezed vacuum (TMSV) [12] as a consistent check of the obtained results in Refs. [13, 14]. The TMSV is a Gaussian state with the covariance matrix [12]

$$\Sigma_{\text{in}} = \frac{1}{2} \begin{pmatrix} \sigma_x \cosh 2r & -\mathbb{1} \sinh 2r \\ -\mathbb{1} \sinh 2r & \sigma_x \cosh 2r \end{pmatrix}. \quad (43)$$

It is worthy to point out that in Ref. [14], the TMSV is feeded into the interferometer after the first beam splitter (BS), whereas in Ref. [13], it is feeded before the first BS. To see the effects of loss on the ideal setup, we calculate the relevant results for both of cases.

For the first case, the input-output relation is given by

$$\begin{aligned} a_1^{\text{out}} &= e^{i\phi} a_1^{\text{in}} \sqrt{\epsilon_1} + v_1 \sqrt{1 - \epsilon_1}, \\ a_2^{\text{out}} &= a_2^{\text{in}} \sqrt{\epsilon_2} + v_2 \sqrt{1 - \epsilon_2}, \end{aligned} \quad (44)$$

where ϵ_1, ϵ_2 represent transmissivity of light beams, and notations a_i, v_i are the i -th ($i = 1, 2$) arm's and its bath mode's annihilation operators, respectively. The average excitation number of the bath is assumed as $\langle v_i^\dagger v_i \rangle = N$. The final state is then characterized by the covariance matrix

$$\Sigma_{\text{out}} = \frac{1}{2} \begin{pmatrix} 0 & d_1 & be^{i\phi} & 0 \\ d_1 & 0 & 0 & be^{-i\phi} \\ be^{i\phi} & 0 & 0 & d_2 \\ 0 & be^{-i\phi} & d_2 & 0 \end{pmatrix}, \quad (45)$$

where $d_i = \epsilon_i \cosh 2r + (1 - \epsilon_i)(2N + 1)$ and $b = -\sqrt{\epsilon_1 \epsilon_2} \sinh 2r$. Substituting it into Eq. (40) yields

$$\mathcal{F}(\phi) = \frac{2b^2}{1 + d_1 d_2 - b^2}, \quad (46)$$

which is independent of ϕ . For $\epsilon_1 = \epsilon_2 = 1$, $\mathcal{F} = \sinh^2 2r$ [14].

For the second case, the input-output relation is

$$\begin{aligned} a_1^{\text{out}} &= e^{i\phi} (a_1^{\text{in}} + a_2^{\text{in}}) \sqrt{\epsilon_1/2} + v_1 \sqrt{1 - \epsilon_1}, \\ a_2^{\text{out}} &= (a_1^{\text{in}} - a_2^{\text{in}}) \sqrt{\epsilon_2/2} + v_2 \sqrt{1 - \epsilon_2}. \end{aligned} \quad (47)$$

The covariance matrix for the final state is

$$\Sigma_{\text{out}} = \frac{1}{2} \begin{pmatrix} b_1 e^{2i\phi} & d_1 & 0 & 0 \\ d_1 & b_1 e^{-2i\phi} & 0 & 0 \\ 0 & 0 & b_2 & d_2 \\ 0 & 0 & d_2 & b_2 \end{pmatrix}, \quad (48)$$

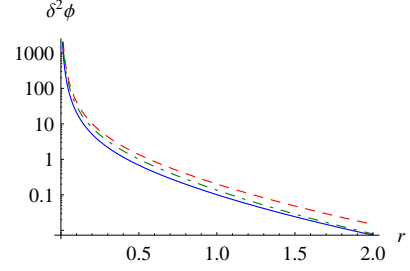


FIG. 1: The plots of $\delta^2 \phi$ versus the squeeze parameter r for the TMSV inserted before (solid) and after (dashed/dot-dashed) the first BS. Here the average excitation number $N = 0.2$. The transmissivity of light beam $\epsilon_1 = 0.8$, $\epsilon_2 = 0.8/1.0$ (dashed/dot-dashed).

where $d_i = \epsilon_i \cosh 2r + (1 - \epsilon_i)(2N + 1)$ and $b_i = -\epsilon_i \sinh 2r$. We can see that, after the first BS, the TMSV becomes disentangled, i.e.

$$S_2(r) \equiv e^{r(a_1 a_2 - a_1^\dagger a_2^\dagger)} \rightarrow e^{r(a_1^2 - a_1^{\dagger 2})/2} e^{r(a_2^2 - a_2^{\dagger 2})/2}, \quad (49)$$

where $S_2(r)$ is the two-mode squeezing operator. The resulting QFI is given by

$$\mathcal{F}(\phi) = \frac{4b_1^2}{1 + d_1^2 - b_1^2}, \quad (50)$$

which becomes $\mathcal{F} = 2 \sinh^2 2r$ for $\epsilon_1 = \epsilon_2 = 1$ [13], twice of that for the first case. For the lossy interferometer, Fig. 1 shows that the quantum entangled state does not always perform better than coherent but disentangled state for phase estimation.

Now let us prove the attainability of the two bounds. By inspecting the structure of Σ_{out} , we take the measurement scheme as $M_{\text{after}} = i(a_1^{\dagger 2} - a_1^2)/2$ and $M_{\text{before}} = i(a_1^\dagger a_2^\dagger - a_1 a_2)/2$, respectively. The phase sensitivities are evaluated,

$$\begin{aligned} \delta^2 \phi &= \frac{\Delta^2 M}{|d\langle M \rangle / d\phi|^2} \\ &= \begin{cases} \frac{1 + d_1 d_2 + b^2(1 - 2 \cos 2\phi)}{2b^2 \cos^2 \phi}, \\ \frac{1 + d_1^2 + b_1^2(1 - 3 \cos 4\phi)/2}{4b_1^2 \cos^2 2\phi}. \end{cases} \end{aligned} \quad (51)$$

The optimization of $\delta^2 \phi$ over ϕ is achieved at $\phi = 0$, which are just Eqs. (46) and (50).

B. Estimation of two conjugate parameters in the displacement operator

Next, we jointly estimate the two conjugate parameters λ_R and λ_I of the displacement operator $D(\lambda) = e^{\lambda a_1^\dagger - \lambda^* a_1}$ with a measurement on the displaced state $\rho = D(\lambda) \rho_0 D^\dagger(\lambda)$ [11]. If we take the two-mode squeezed

thermal state $\rho_0 = S_2(r)(\rho_{\nu_T} \otimes \rho_{\nu_T})S_2^\dagger(r)$ as the input, where

$$\rho_{\nu_T} = \frac{1}{\nu_T + 1} \sum_{n=0}^{\infty} \left(\frac{\nu_T}{\nu_T + 1} \right)^n |n\rangle\langle n| \quad (52)$$

is a single-mode thermal state with average excitation number ν_T , the mean displacement and covariance matrix of this Gaussian state are given by $\lambda_{\text{in}} = 0$ and

$$\Sigma_{\text{in}} = \frac{2\nu_T + 1}{2} \begin{pmatrix} \sigma_x \cosh 2r & -\mathbb{1} \sinh 2r \\ -\mathbb{1} \sinh 2r & \sigma_x \cosh 2r \end{pmatrix}. \quad (53)$$

If we include the possible photon loss described by Eqs. (44) with $\phi = 0$ before the displacement operator, the final state is characterized by $\lambda_{\text{out}} = (\lambda, \lambda^*, 0, 0)^T$, and

$$\Sigma_{\text{out}} = \frac{1}{2} \begin{pmatrix} d_1 \sigma_x & b \mathbb{1} \\ b \mathbb{1} & d_2 \sigma_x \end{pmatrix}, \quad (54)$$

where $d_i = \epsilon_i(2\nu_T + 1) \cosh 2r + (1 - \epsilon_i)(2N + 1)$ and $b = -\sqrt{\epsilon_1 \epsilon_2}(2\nu_T + 1) \sinh 2r$. The two bounds can be straightforwardly evaluated from Eqs. (7), (8), (35), and (40), obtaining

$$\begin{aligned} B_R &= \frac{d_1}{2} + \frac{d_2 b^2}{2(1 - d_2^2)} + \left| \frac{1}{2} + \frac{b^2}{2(1 - d_2^2)} \right|, \\ B_S &= \frac{d_1}{2} - \frac{b^2}{2d_2}. \end{aligned} \quad (55)$$

For lossless case $\epsilon_1 = \epsilon_2 = 1$, they are in agreement with those in Ref. [11]. Using the same homodyne measurement scheme proposed in Ref. [11], the sum the two resulting variances is $B_M = (d_1 + d_2)/2 - b$.

Fig. 2 displays the three bounds $B_{R,S,M}$ versus the squeezing parameter r . We see that which bound is tighter depends on the actual values of r , and the bound B_M from the homodyne measurement is always higher than the theoretical RLD and SLD bounds. We also note that B_M stays much closer to the theoretical bound for the balanced losses than for the unbalanced losses in the two arms.

C. Estimation of damping and temperature

Then, we consider the problem of estimating the parameters of a Gaussian channel [6] describing the evolution of a bosonic mode a_1 , coupled with strength γ to a thermal bath mode v_1 with mean excitation number N . The completely positive dynamics of the mode a_1 in the interaction frame under the Markovian approximation is represented by the unitary transformation

$$a_1^{\text{out}} = a_1^{\text{in}} e^{-\gamma/2} + v_1 \sqrt{1 - e^{-\gamma}}. \quad (56)$$

We first take the single-mode state parameterized by

$$\rho_{\text{in}} = D(\lambda) S(r) \rho_{\nu_T} S^\dagger(r) D^\dagger(\lambda) \quad (57)$$

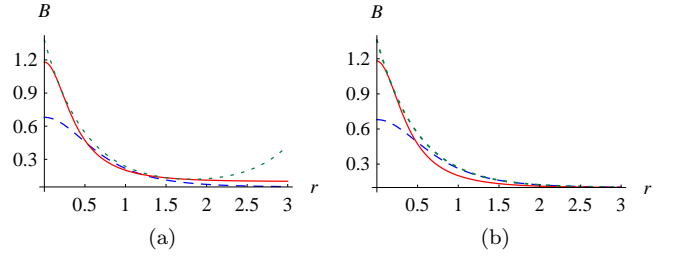


FIG. 2: The plots of B versus the squeeze parameter r for the estimation of position and momentum. (a) $\epsilon_1 = 0.9$, $\epsilon_2 = 1.0$ for the unbalanced losses in two arms. (b) $\epsilon_1 = \epsilon_2 = 0.9$ for the balanced losses in two arms. Here the solid (dashed) lines are for the RLD (SLD) bound, and the dotted lines represent the bound from the homodyne measurement. The other parameters are $\nu_T = 0.2$ and $N = 0$.

as the input, where the single-mode squeezing operator $S(r) = e^{r(a^2 - a^{\dagger 2})/2}$. The mean displacement and covariance matrix of this state are given by $\lambda_{\text{in}} = (\lambda, \lambda^*)^T$ and

$$\Sigma_{\text{in}} = \frac{2\nu_T + 1}{2} \begin{pmatrix} -\sinh 2r & \cosh 2r \\ \cosh 2r & -\sinh 2r \end{pmatrix}. \quad (58)$$

The final state would be described by $\lambda_{\text{out}} = (e^{-\gamma/2}\lambda, e^{-\gamma/2}\lambda^*)^T$ and

$$\Sigma_{\text{out}} = \frac{1}{2} \begin{pmatrix} b & d \\ d & b \end{pmatrix}, \quad (59)$$

where $b = -e^{-\gamma}(2\nu_T + 1) \sinh 2r$ and $d = e^{-\gamma}(2\nu_T + 1) \cosh 2r + (1 - e^{-\gamma})(2N + 1)$. Since the final expressions for the QFI matrices are too lengthy, we only display them numerically in Fig. 3 (a). We see that the SLD bound is tighter than the RLD bound, as the asymptotic attainability condition (9) is fulfilled here.

If we take the TMSV (a_2 being the ancillary mode) as the input with the mean displacement $\lambda_{\text{in}} = (\lambda_1, \lambda_1^*, \lambda_2, \lambda_2^*)$ and covariance matrix (43), the final state will be described by $\lambda_{\text{out}} = (e^{-\gamma/2}\lambda_1, e^{-\gamma/2}\lambda_1^*, \lambda_2, \lambda_2^*)$, and

$$\Sigma_{\text{out}} = \frac{1}{2} \begin{pmatrix} d_1 \sigma_x & b \mathbb{1} \\ b \mathbb{1} & d_2 \sigma_x \end{pmatrix}, \quad (60)$$

where $d_1 = e^{-\gamma} \cosh 2r + (1 - e^{-\gamma})(2N + 1)$, $b = -e^{-\gamma/2} \sinh 2r$, and $d_2 = \cosh 2r$.

Noting the form of the covariance matrix and the symmetry of Eq.(38), the coefficient matrix \mathcal{A} of SLD takes the form

$$\mathcal{A}^{(k)} = \frac{1}{2} \begin{pmatrix} d_1^{(k)} \sigma_x & b^{(k)} \mathbb{1} \\ b^{(k)} \mathbb{1} & d_2^{(k)} \sigma_x \end{pmatrix}, \quad k = \gamma, N. \quad (61)$$

Actually we do not even know the explicit expression for the SLD to prove the asymptotically attainability of the SLD bound in such a case, but recall Eqs. (42), (60),

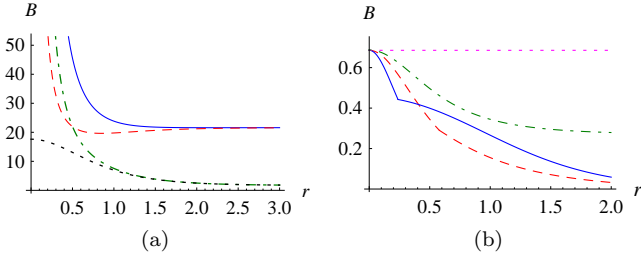


FIG. 3: (a) The plots of B versus the squeeze parameter r for the estimation of damping and temperature. The solid/dashed (dot-dashed/dotted) line is for the B_S/B_R with the single-mode squeezed (TMSV) state. Here $N = 0.9$, $\xi = 0.5$, $\lambda = 0$, and $\nu_T = 0$. (b) The plots of $B = \max\{B_R, B_S\}$ versus the probe energy n (expressed in terms of the effective squeeze parameter r via $n = (\nu_T + 1/2) \cosh 2r - 1$) for the estimation of squeezing and phase. The solid/dot-dashed (dotted) line is for the single-mode coherent/thermal (squeezed) state. The dashed line is for the two-mode squeezed thermal state. Here $s = 1$, $\varphi = 0$, and $\nu_T = 0.1$.

(61) and the fact of $\mathcal{B}^{(N)} = 0$ implied by $\partial_N \lambda_{\text{out}} = 0$. Eqs.(35) and (40) give the results

$$\mathcal{F}_{\gamma\gamma} = \frac{1}{\xi^2} + \frac{\xi t + 2}{8Y\xi^2}, \quad \mathcal{F}_{NN} = \frac{1}{Y}, \quad \mathcal{F}_{\gamma N} = \frac{y}{2Y\xi}, \quad (62)$$

and

$$\mathcal{F}_{\gamma\gamma} = \frac{2\xi n(n+1) + t - 2}{\xi(\xi t + 2)}, \quad (63)$$

$$\mathcal{F}_{NN} = \frac{\xi t}{Y(\xi t + 2)}, \quad \mathcal{F}_{\gamma N} = \frac{2(2n+1)}{\xi t + 2},$$

where $\xi = e^\gamma - 1$, $n = \sinh^2 r$, $y = 2N + 1$, $t = y(2n + 1) + 1$, and $Y = N(N + 1)$. It is verified that Eqs. (63) always give a tighter bound than Eqs. (62), as indicated by Fig. 3 (a), in comply with the asymptotic attainability of the SLD bound. We also note that the TMSV gives better precision than the single-mode state.

It is also noticed that when the damping is extremely small $\xi \rightarrow 0$, no information of temperature is gained, i.e. $\delta N \rightarrow \infty$. On the other hand, when the damping is extremely large $\xi \rightarrow \infty$, or the output state is in equilibrium with the thermal bath, no information of damping is gained, i.e. $\delta\gamma \rightarrow \infty$, and $\delta N \rightarrow 1/Y$. By contrast, there might exit some typos for Eqs. (B9a)-(B9d) in Ref. [6], which contain the erroneous extra factor of $e^{2\gamma}$.

D. Estimation of squeezing and phase

Finally, we address the estimation of $\theta = (s, \varphi)^T$, i.e. the squeezing and phase parameters in $\zeta = se^{2i\varphi}$ of the squeezing operator $S(\zeta) = e^{(\zeta^* a_1^2 - \zeta a_1^{\dagger 2})/2}$ with a measurement on the state $\rho_{\text{out}} = S(\zeta)\rho_{\text{in}}S^\dagger(\zeta)$. In order to determine the precision attainable with different Gaussian states, we first take the single-mode state param-

eterized by Eq. (57) as the input. The mean displacement and covariance matrix of the final state can be evaluated by the relations $D^\dagger(\lambda)a_1D(\lambda) = a_1 + \lambda$ and $S^\dagger(\zeta)a_1S(\zeta) = a_1 \cosh s - a_1^\dagger e^{i\varphi} \sinh s$. From Eqs. (35) and (40), the RLD and SLD QFI matrices are obtained,

$$\mathcal{F}_{\varphi\varphi} = \frac{y^2(2Y+1)}{2Y^2} \sinh^2 2s + \frac{4y}{Y} |\lambda|^2 e^{-2r-2s} \sinh^2 s,$$

$$\mathcal{F}_{ss} = \frac{y^2(2Y+1)}{2Y^2} + \frac{y}{Y} |\lambda|^2 e^{2r},$$

$$\mathcal{F}_{s\varphi} = i \left(\frac{y^3}{2Y^2} \sinh 2s + \frac{2}{Y} |\lambda|^2 e^{-s} \sinh s \right), \quad (64)$$

and

$$\mathcal{F}_{\varphi\varphi} = \frac{2y^2}{2Y+1} \sinh^2 2s + \frac{16}{y} |\lambda|^2 e^{-2r-2s} \sinh^2 s,$$

$$\mathcal{F}_{ss} = \frac{2y^2}{2Y+1} + \frac{4}{y} |\lambda|^2 e^{2r}, \quad \mathcal{F}_{s\varphi} = 0, \quad (65)$$

where $y = 2\nu_T + 1$, $Y = \nu_T(\nu_T + 1)$, and the phase parameter is assumed around $\varphi = 0$ for simplicity. In order to compare the performances for the estimation, we consider the three different situations: coherent state ($r = 0$), squeezed state ($\lambda = 0$), and thermal state ($r = \lambda = 0$). The respective energies are given by $n = |\lambda|^2 + \nu_T$, $n = (\nu_T + 1/2) \cosh 2r - 1/2$, and $n = \nu_T$.

Now we turn to the two-mode squeezed thermal state $\rho_{\text{in}} = S_2(r)(\rho_{\nu_T} \otimes \rho_{\nu_T})S_2^\dagger(r)$ (a_2 being the ancillary mode) as the input, and its energy is $n = (\nu_T + 1/2) \cosh 2r - 1/2$. The QFI matrices can also be calculated from Eqs. (64). The full expressions for \mathcal{F} and \mathcal{F} are too lengthy to put here, we only display the numerical results in Fig. 3 (b). It can be seen that for the single-mode states, the coherent state gives the best performance among others, and the squeezed state is even worse than the thermal state. The two-mode squeezed thermal state performs better than the coherent state only when the probe energy is larger than some actual value. Moreover, the RLD bounds are more tighter than the SLD bounds when the probe energies are relatively lower.

V. CONCLUSION

In this paper, we have studied the quantum Cramer-Rao bounds on the joint multiple-parameter estimation with the Gaussian state as a probe. We have derived the explicit forms of the right logarithmic derivative and symmetric logarithmic derivative operators for the Gaussian state. We have also calculated the corresponding quantum Fisher information matrices, and found that they can be fully expressed in terms of the mean displacement and covariance matrix of the Gaussian state. We have taken some explicit examples to show the utility of our analytical results.

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